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MIMO2: Capacity

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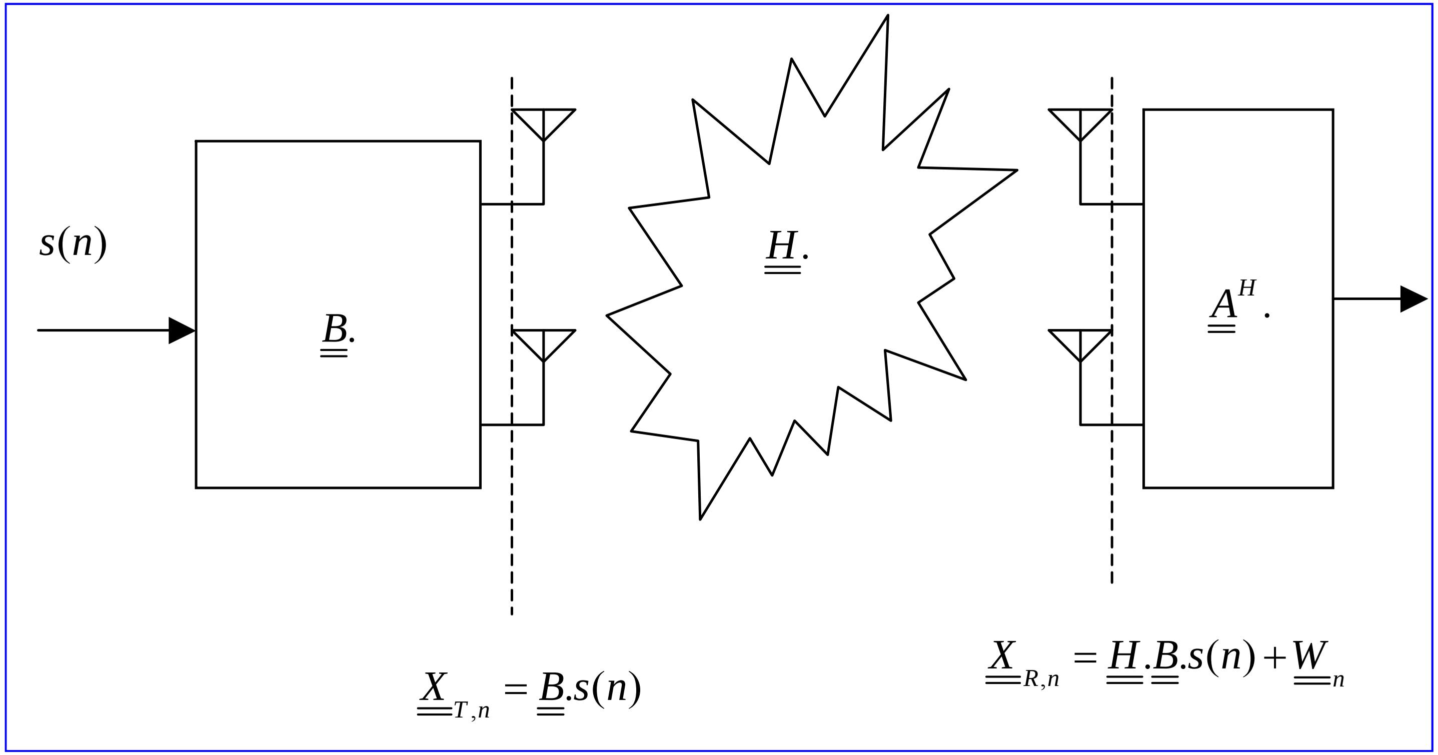
When RATE is important



$$P_e = Q\left(\sqrt{\frac{2.E_T}{N_0}.\lambda_{\max}\left(\underline{\underline{R}}_H\right).\left(\frac{3}{\left(2^{n_S}-1\right)}\right)}\right)$$

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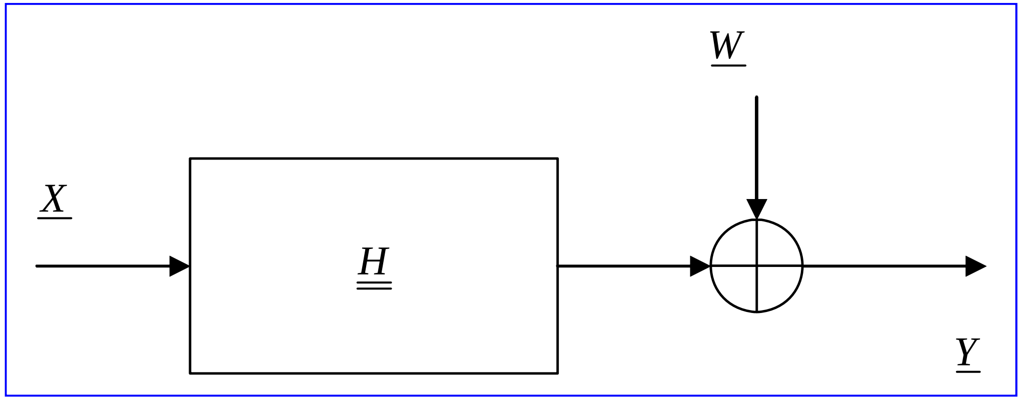
Communications is a problem of MUTUAL INFORMATION



The mutual information between two random vectors is:

$$I(\underline{X},\underline{Y}) = \iint f(\underline{X},\underline{Y}).Ln\left[\frac{f(\underline{X},\underline{Y})}{f(\underline{X}).f(\underline{Y})}\right].d\underline{X}.d\underline{Y}$$

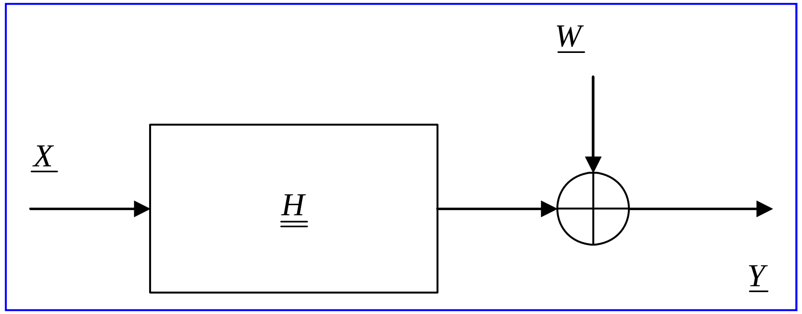
Maximice I(X,Y) with respect noise and channel

ONLY Tx has something to do with it!!!

Maximization: First steep

Assume that:

$$\underline{X} = G\left(\underline{0}, \underline{\underline{Q}}\right) \quad \underline{W} = G\left(\underline{0}, \underline{\underline{R}}_0\right)$$



1.-Gaussian Coding: The first steep to maximize I(X,Y)

$$\max_{f(.)} \int f(x).Ln\left(f(x)\right).dx$$

$$\int f(x).dx = 1$$

$$\int x^2.f(x).dx = \sigma_x^2$$

$$\longrightarrow \quad f(x) = k.e^{-\frac{x^2}{2.\sigma^2}}$$

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Second steep_ Pass I : The Importance of the Determinant of the Tx Covariance

The input distribution and Tx power

$$\Pr\left(\underline{X}\right)=\det\left(\underline{\underline{Q}}^{-1}\right).\exp\left\{-\left(\underline{X}^{H}.\underline{\underline{Q}}.\underline{X}\right)\right\}$$

Two terms for the Mutual Information

Joint distribution

$$I=\iint f\left(\underline{X},\underline{Y}\right).Ln\left[\frac{\det\left[\underline{\underline{Q}}\right].\det\left[\underline{\underline{R}}_{yy}\right]}{\det\begin{pmatrix}\underline{\underline{Q}} & \underline{\underline{R}}_{xy}\\ \underline{\underline{R}}_{yx} & \underline{\underline{R}}_{yy}\end{pmatrix}}\right].d\underline{X}.d\underline{Y}+$$

$$-\iint f\left(\underline{X},\underline{Y}\right).\left[\underline{X}^{H}.\underline{\underline{R}}_{xy}.\underline{Y}+\underline{Y}^{H}.\underline{\underline{R}}_{yx}.\underline{X}\right].d\underline{X}.d\underline{Y}$$

Cross-terms equal to cero

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$$I(\underline{X},\underline{Y}) = Ln\left[\frac{\det\left[\underline{\underline{Q}}\right].\det\left[\underline{\underline{R}}_{yy}\right]}{\det\begin{pmatrix}\underline{\underline{Q}} & \underline{\underline{R}}_{xy}\\ \underline{\underline{R}}_{yx} & \underline{\underline{R}}_{yy}\end{pmatrix}}\right]$$

Using the determinant rule (Schur)

$$\det\begin{pmatrix}\underline{\underline{Q}} & \underline{\underline{R}}_{xy}\\ \underline{\underline{R}}_{yx} & \underline{\underline{R}}_{yy}\end{pmatrix} = \det\left[\underline{\underline{Q}}\right].\det\left[\underline{\underline{R}}_{yy} - \underline{\underline{R}}_{yx}.\underline{\underline{Q}}^{-1}.\underline{\underline{R}}_{xy}\right]$$

$$C = Ln\left[\frac{\det\left[\underline{\underline{R}}_{yy}\right]}{\det\left[\underline{\underline{R}}_{yy} - \underline{\underline{R}}_{yx}.\underline{\underline{Q}}^{-1}.\underline{\underline{R}}_{xy}\right]}\right]$$

Received Signal

Received Noise plus Intereference

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MI in terms of channel, noise and Tx covariance

$$C = Ln\left[\frac{\det\left[\underline{\underline{R}}_{yy}\right]}{\det\left[\underline{\underline{R}}_{yy} - \underline{\underline{R}}_{yx}.\underline{\underline{Q}}^{-1}.\underline{\underline{R}}_{xy}\right]}\right]$$

with

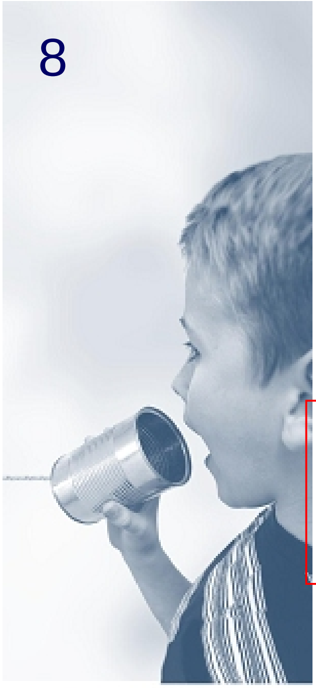
$$\underline{\underline{R}}_{yy} = \underline{\underline{H}}.\underline{\underline{Q}}.\underline{\underline{H}}^{H} + \underline{\underline{R}}_{0}$$

$$\underline{\underline{R}}_{xy} = \underline{\underline{Q}}.\underline{\underline{H}}^{H}$$

$$\underline{\underline{R}}_{yy} = \underline{\underline{H}}.\underline{\underline{Q}}$$

We denote MI as Capacity C but it is not strictly capacity

$$C = Ln\left[\frac{\det\left[\underline{\underline{R}}_{0} + \underline{\underline{H}}.\underline{\underline{Q}}.\underline{\underline{H}}^{H}\right]}{\det\left[\underline{\underline{R}}_{0}\right]}\right] = Ln\left(\det\left[\underline{\underline{I}}_{n_R} + \underline{\underline{R}}_{0}^{-1}\underline{\underline{H}}.\underline{\underline{Q}}.\underline{\underline{H}}^{H}\right]\right)$$



MI: Engineering terms

$$C = Ln\left(\det\left[\underline{\underline{I}}_{n_R} + \underline{\underline{R}}_0^{-1}\underline{\underline{H}}.\underline{\underline{Q}}.\underline{\underline{H}}^H\right]\right) = Ln\left(\det\left[\underline{\underline{I}}_{n_R} + \underline{\underline{H}}^H\underline{\underline{R}}_0^{-1}\underline{\underline{H}}.\underline{\underline{Q}}\right]\right)$$

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$$\underline{\underline{R}}_H = \underline{\underline{H}}^H.\underline{\underline{R}}_0^{-1}.\underline{\underline{H}}$$

The channel SNR

$$\underline{\underline{R}}_H.\underline{\underline{Q}} = \underline{\underline{H}}^H.\underline{\underline{R}}_0^{-1}.\underline{\underline{H}}.\underline{\underline{Q}}$$

The absolute SNR of the communications scenario

Let us see some examples for MIMO channels

MI (Capacity): Some Examples

Capacity for Uniform Power Allocation UPA

$$C = Ln\left(\det\left[\underline{\underline{I}}_{n_R} + \frac{\gamma}{n_T}\underline{\underline{H}}.\underline{\underline{H}}^H\right]\right)$$

with

$$\gamma = \frac{E_T}{N_0}$$

Normaliced with theFroebenius norm of the trace

$$C_{SISO} = Ln\left[1+\gamma\right]$$

$$C_{MISO} = Ln\left[1+\frac{\gamma}{n_T}\underline{h}^H.\underline{h}\right]$$

$$C_{SIMO} = Ln\left[1+\gamma.\underline{h}^H.\underline{h}\right]$$

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No advantage for capacity derived for increasing either the Tx number of antennas (MISO) or the Rx number of antennas (SIMO)

$$C_{MISO} = Ln\left[1 + \frac{\gamma}{n_T}\underline{h}^H . \underline{h}\right]$$

$$C_{SIMO} = Ln\left[1 + \gamma . \underline{h}^H . \underline{h}\right]$$

$$C_{MISO} = C_{SIMO} = Ln[1+\gamma]$$

$$C_{MISO} = Ln[1+\gamma]$$

$$C_{SIMO} = Ln[1+\gamma . n_R]$$

MIMO Increase linearly the channel capacity

Assuming UPA for the channel eigenmodes, the mutual information is (sub-optimum):

$$C = \sum_{q=1}^{\min(n_T, n_R)} Ln\left[1 + \frac{E_T}{N_0.n_T}.\lambda_q\right]$$

And, for uniform channel eigenvalues (gain), MI scales linearly with the minimum number of antennas among Tx/Rx (Worst channel case).

$$C = \min(n_T, n_R).Ln\left[1 + \frac{E_T.\lambda}{n_T.N_0}\right]$$

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Achieving capacity: Step 2.- Optimal Tx architecture

$$C = Ln\left(\det\left[\underline{\underline{I}}_{n_T} + \underline{\underline{Q}}.\underline{\underline{H}}^H \underline{\underline{R}}_0^{-1} \underline{\underline{H}}\right]\right) = Ln\left(\det\left[\underline{\underline{I}}_{n_R} + \underline{\underline{H}}^H \underline{\underline{R}}_0^{-1} \underline{\underline{H}}.\underline{\underline{Q}}\right]\right) =$$

$$C = Ln\left(\det\left[\underline{\underline{I}}_{n_T} + \underline{\underline{Q}}.\underline{\underline{R}}_H\right]\right)$$

Since, for definite positive matrixes

$$\det\left(\underline{\underline{A}}\right) \leq \prod A_{ii}$$

The optimal Tx strategy is to diagonalice the channel matrix

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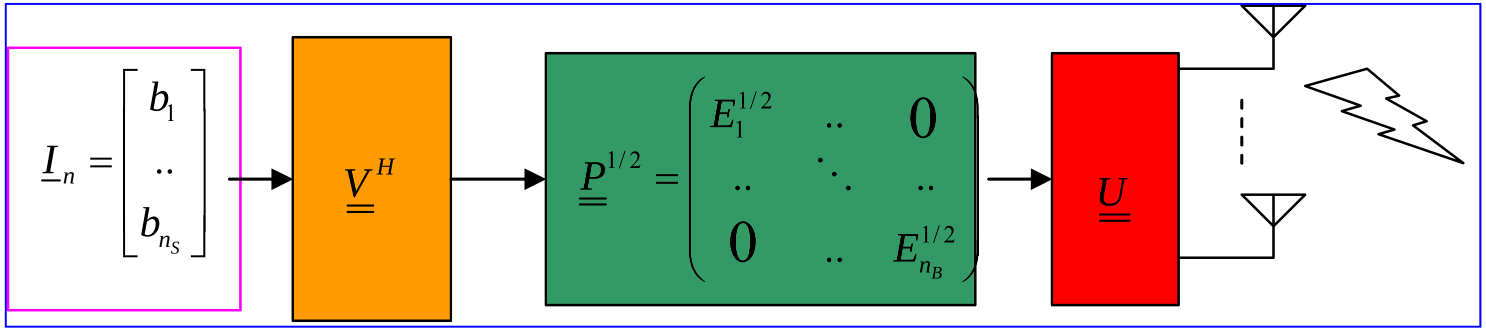
With the svd of the channel matrix

$$\underline{\underline{R}}_H = \underline{\underline{U}}.\underline{\underline{L}}.\underline{\underline{U}}^H = \left[\underline{u}_{max} \quad \cdots \quad \underline{u}_{min}\right].\begin{bmatrix} \lambda_{max} & .. & 0 \\ .. & .. & .. \\ 0 & .. & \lambda_{min} \end{bmatrix}.\begin{bmatrix} \underline{u}_{max}^H \\ .. \\ \underline{u}_{min}^H \end{bmatrix}$$



$$\underline{\underline{Q}} = \underline{\underline{B}}.\underline{\underline{B}}^H = \left(\underline{\underline{U}}.\underline{\underline{P}}^{1/2}.\underline{\underline{V}}^H\right).\left(\underline{\underline{U}}.\underline{\underline{P}}^{1/2}.\underline{\underline{V}}^H\right)^H = \underline{\underline{U}}.\underline{\underline{P}}.\underline{\underline{U}}^H$$

MI independent of matrix V, only beamforming as the eigenvectors of R_H is required

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$$I(X,Y) = Ln\left(\det\left[\underline{\underline{I}}_{n_T} + \frac{1}{N_0}\underline{\underline{U}}_H.\underline{\underline{P}}^{1/2}.\underline{\underline{V}}^H.\underline{\underline{V}}.\underline{\underline{P}}^{1/2}.\underline{\underline{D}}.\underline{\underline{U}}_H^H\right]\right) =$$

$$I(X,Y) = Ln\left(\det\left[\underline{\underline{I}}_{\min(n_T,n_R)} + \frac{1}{N_0}\underline{\underline{P}}^{1/2}.\underline{\underline{V}}^H.\underline{\underline{V}}.\underline{\underline{P}}^{1/2}.\underline{\underline{D}}_H\right]\right)$$

$$\underline{\underline{V}}^H.\underline{\underline{V}} = \underline{\underline{I}}_{n_T} \qquad \underline{\underline{Q}} = \underline{\underline{U}}.\underline{\underline{Z}}.\underline{\underline{U}}^H \qquad \underline{\underline{R}}_H = \underline{\underline{U}}.\underline{\underline{L}}.\underline{\underline{U}}^H$$

$$I\left(\underline{X},\underline{Y}\right) = C = \ln\left[\det\left(\underline{\underline{I}} + \underline{\underline{Z}}.\underline{\underline{L}}\right)\right]$$

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Capacity: Steep 3.- Optimum power allocation

$$C = Ln\left(\det\left[\underline{\underline{I}}_{\min(n_T,n_R)} + \frac{1}{N_0}\underline{\underline{P}}^{1/2}.\underline{\underline{V}}^{H}.\underline{\underline{V}}.\underline{\underline{P}}^{1/2}.\underline{\underline{D}}_{H}\right]\right) =$$

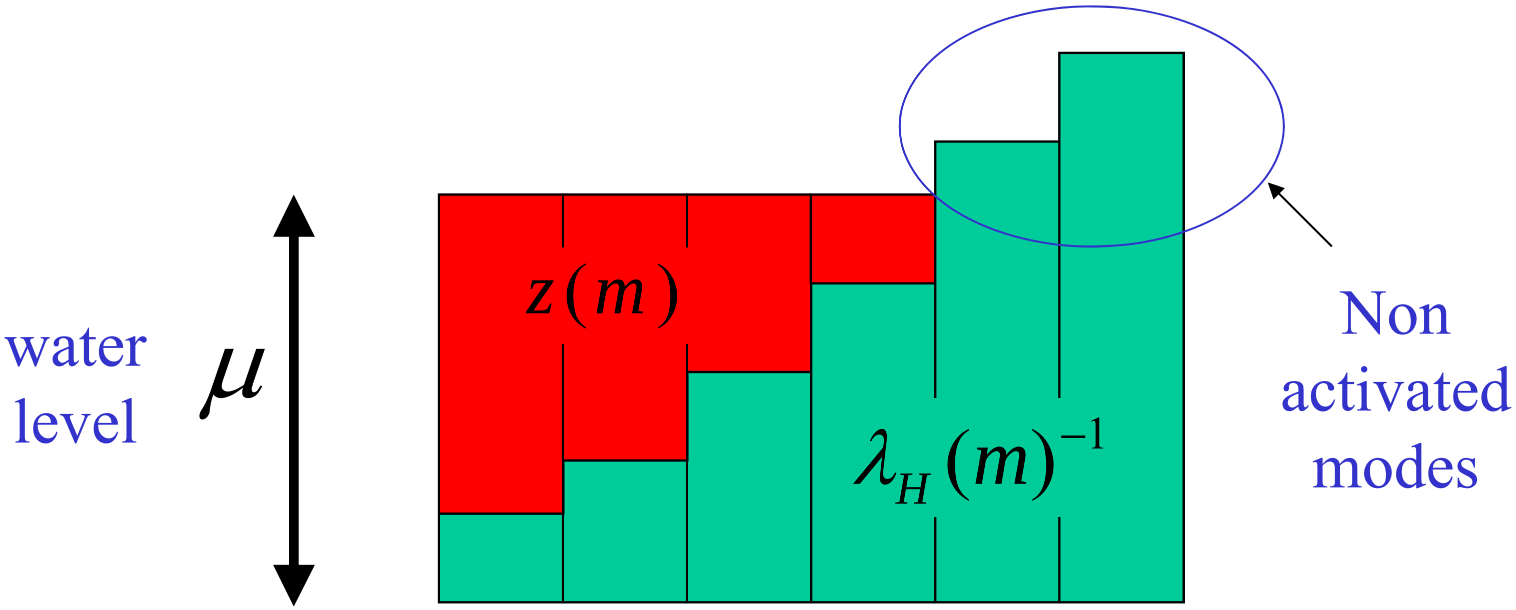
$$= \sum_{q=1}^{\min(n_T,n_R)} Ln\left[1 + \frac{1}{N_0} z(q).\lambda_H(q)\right]\Bigg|_{MAX}$$

Constrained to $\sum_{q=1}^{\min(n_T,n_R)} z(q) = E_T$ The Lagrangian is:

$$\Delta = \sum_{q=1}^{\min(n_T,n_R)} Ln\left[1 + \frac{1}{N_0} z(q).\lambda_H(q)\right] - m1.\left(\sum_{q=1}^{\min(n_T,n_R)} z(q) = E_T\right)$$

$$\partial\Delta = 0 = \frac{1}{N_0}\frac{\lambda_H(q)}{1 + z(q).\lambda_H(q)} - m1 \quad \Rightarrow \quad z(q) = \left[\mu - \frac{1}{\lambda_H(q)}\right]^{+}$$

Water-filling algorithm



- Note that not all the modes are used

-Fairness issue: More energy for the best modes

-Gaussian coding and bit-loading requiered

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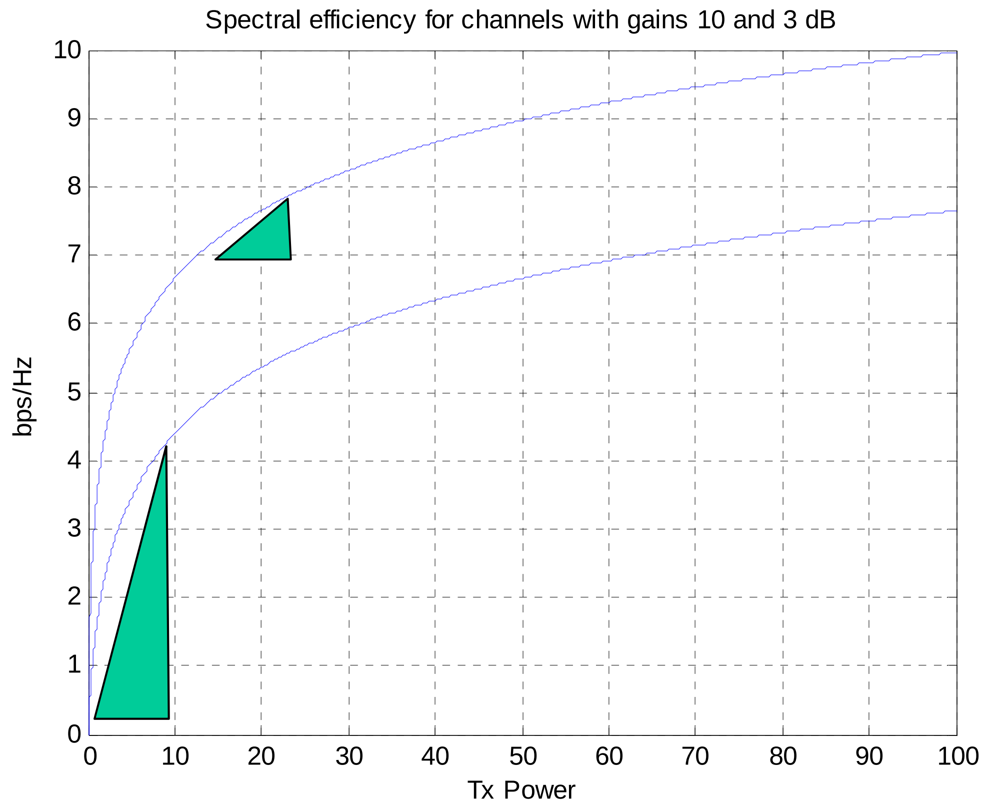
An intuitive explanation



Due to saturation of log(.) it is more rewerding to start allocating power to the lower channel

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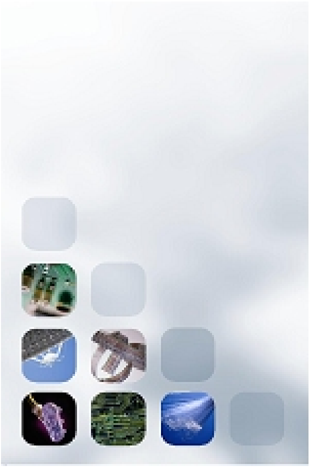

Capacity for UPA

When UPA is used the capacity is:

$$C = \ln\left(\det\left[\underline{\underline{I}} + \frac{E_T}{n_T}\underline{\underline{R}}_H\right]\right) = \sum_{q=1}^{\min(n_T,n_R)} \ln\left(1 + \frac{E_T.\lambda_H(q)}{n_T}\right)$$

There is an interesting approximation of this expression for moderate/high SNR regime, showing that the geometric mean of the eigenvalues determines the UPA capacity:

$$C = \ln\left(\prod_{q=1}^{n_M \triangleright \min(n_T,n_R)} \left(1 + \frac{E_T.\lambda_H(q)}{n_T}\right)\right) \approx n_R.\ln\left(\frac{E_T.\lambda_{GEO}}{n_T}\right)$$

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Instantaneous Max. Capacity

When all the modes are activated:

$$\mu = \frac{E_T}{n_M} + \lambda_{HARM} \quad where \quad \lambda_{HARM} = \frac{n_M}{\sum_{q=1}^{n_M} \lambda_H^{-1}(q)}$$

The power allocation is:

$$z(q) = \frac{E_T}{n_M} + \lambda_{HARM}^{-1} - \lambda_H^{-1}(q)$$

Note that all the modes are activated when the mean energy per mode is greater than the difference between the inverse of the harmonic mean and the inverse of the minimum eigenmode gain.

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$$C = n_M . \ln\left(\left[\frac{E_T . \lambda_{GEO}}{n_M}\right] + \left[\frac{\lambda_{GEO}}{\lambda_{HARM}}\right]\right)$$

UPA contribution

(No CSIT required)

Gain derived from CSIT

-The gain derived from CSIT does not depends on the power budget.

- For large SNR regimes the optimum power allocation to achieve capacity is UPA.

- DSL services. Not power limited. No mask constrains (twisted pairs). UPA

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Beamforming to achieve capacity

$$\mu = E_T + \frac{1}{\lambda_{H,\max}} \; ; \; z(0) = E_T$$

$$C_{BEAMFORMING} = Ln\left[1 + \frac{E_T}{N_0} . \lambda_{H,\max}\right]$$

When only the maximum gain mode is activated the optimum policy is beamforming

This is the case when the second and the first eigenmode gains satisfy:

$$\lambda_2 \leq \frac{1}{E_T + \left(\frac{1}{\lambda_1}\right)} \Rightarrow E_T < \left(\frac{1}{\lambda_2}\right) - \left(\frac{1}{\lambda_1}\right)$$

Maximum quality and maximum rate policies coincide

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Example:

$\underline{\underline{H}}^H \underline{\underline{H}}$	$MaxEigenvalue$	Determinant
$\begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}$	$10 \quad (10dB)$	$50 \quad (5.67bps/Hz.)$
$\begin{pmatrix} 8 & 0 \\ 0 & 7 \end{pmatrix}$	$8 \quad (9dB)$	$15 \quad (5.83bps/Hz.)$
$\begin{pmatrix} 10 & 2 \\ 2 & 5 \end{pmatrix}$	$10.70 \quad (10.29dB)$	$46 \quad (5.52bps/Hz.)$
$\begin{pmatrix} 10 & 7 \\ 7 & 5 \end{pmatrix}$	$14.93 \quad (11.7dB)$	$1 \quad (2bps/Hz.)$

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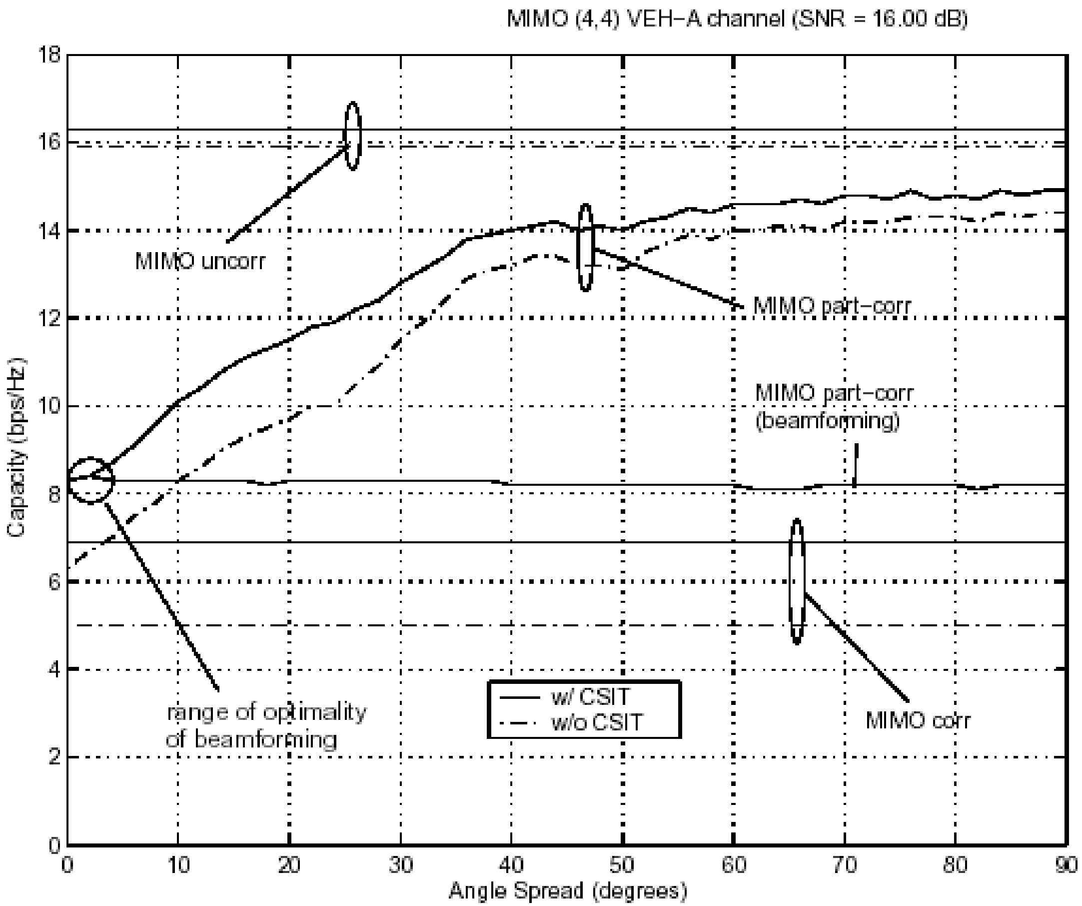
Correlation degrades the geommetric mean (the channel determinant)



Ergodic Capacity

When there is not CSIT, the Tx design depends on the statistical knowledge of the channel and the instantaneous capacity is averaged over channel statistics.

$$\underline{\underline{Q}} = \max_{\underline{\underline{Q}}}\left(E\left[\ln\left(\det\left(\underline{\underline{I}} + \underline{\underline{R}}_H . \underline{\underline{Q}}\right)\right)\right]\right)$$

with $\quad Trace\left(\underline{\underline{Q}}\right) = E_T \quad$ Is the problem to be solved

The solution to the previous problem for channel matrixes with i.i.d. entries following a pdf p(x) such that it is symmetric along the origin, i.e. p(x)=p(-x), and circular symmetry $p(x)=p(x.e^{j\phi})$ is UPA (Uniform Power Allocation)

The optimum policy for maximizing ergodic capacity is UPA

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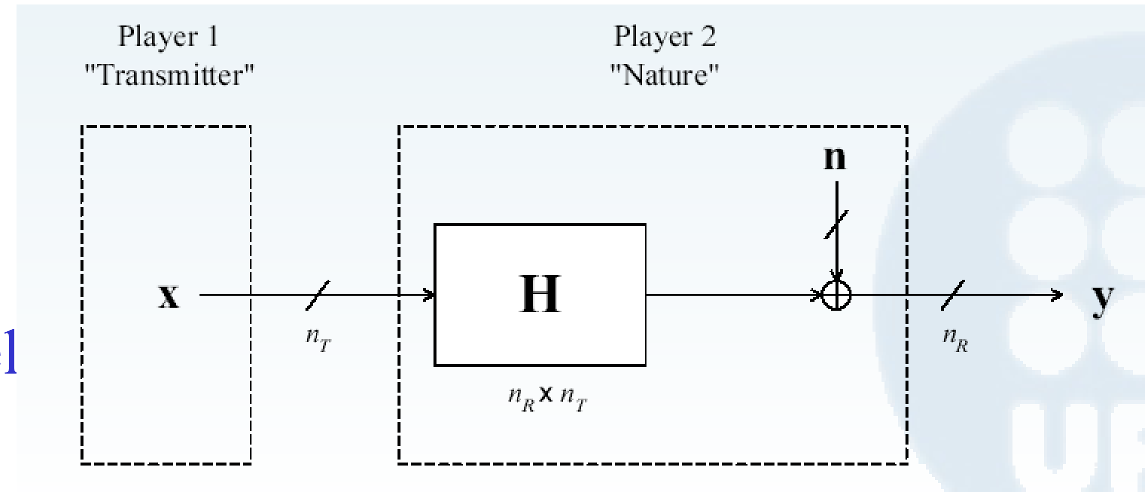
GAME THEORY APPROACH

-Strategic game with pure strategies

-Strategic game with mixed pure strategies (random over pure strategies)

- Stackelberg game: Strategies depends on the previous selected strategy by another player)

Solution independent of the channel constrains



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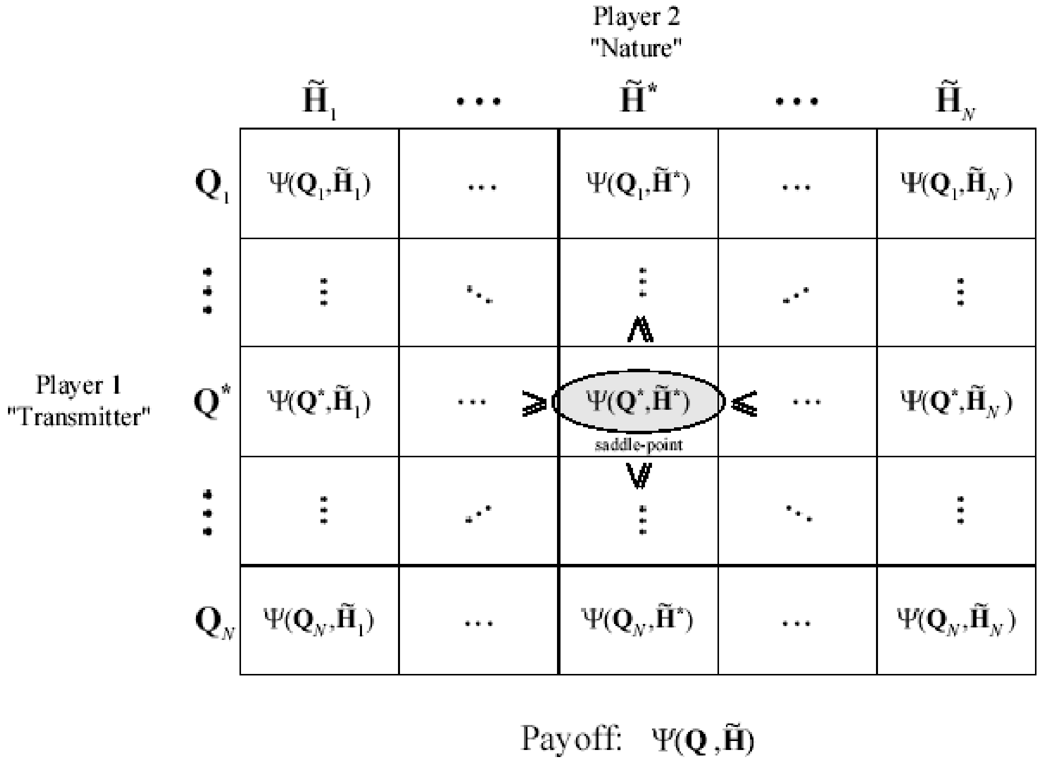

Pure strategies:

-Tx performs water-filling of channel

-Channel performs inverse waterfilling of Tx

- Nash equilibrium for flat Tx and Channel

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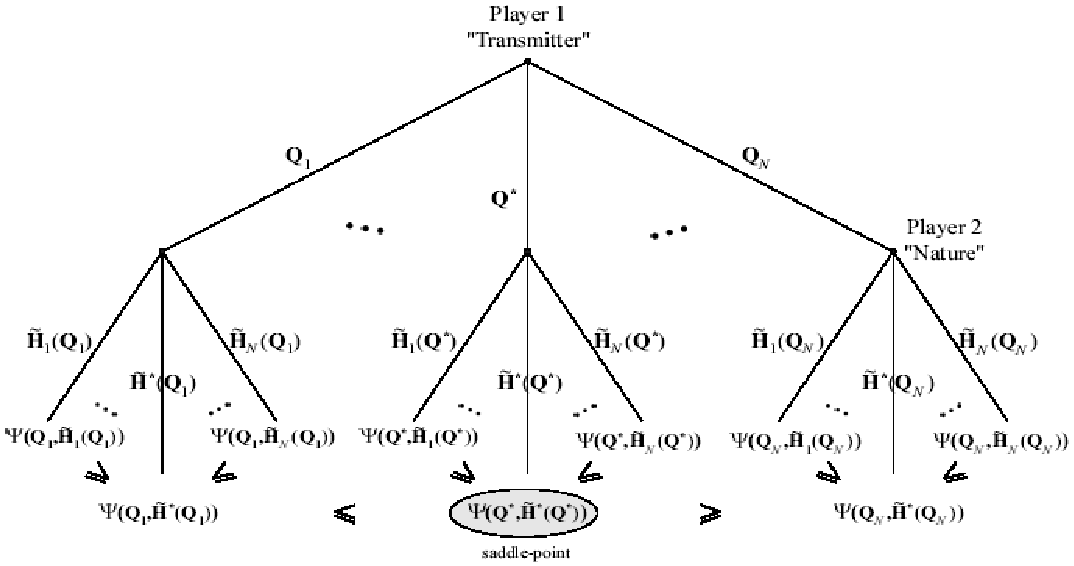
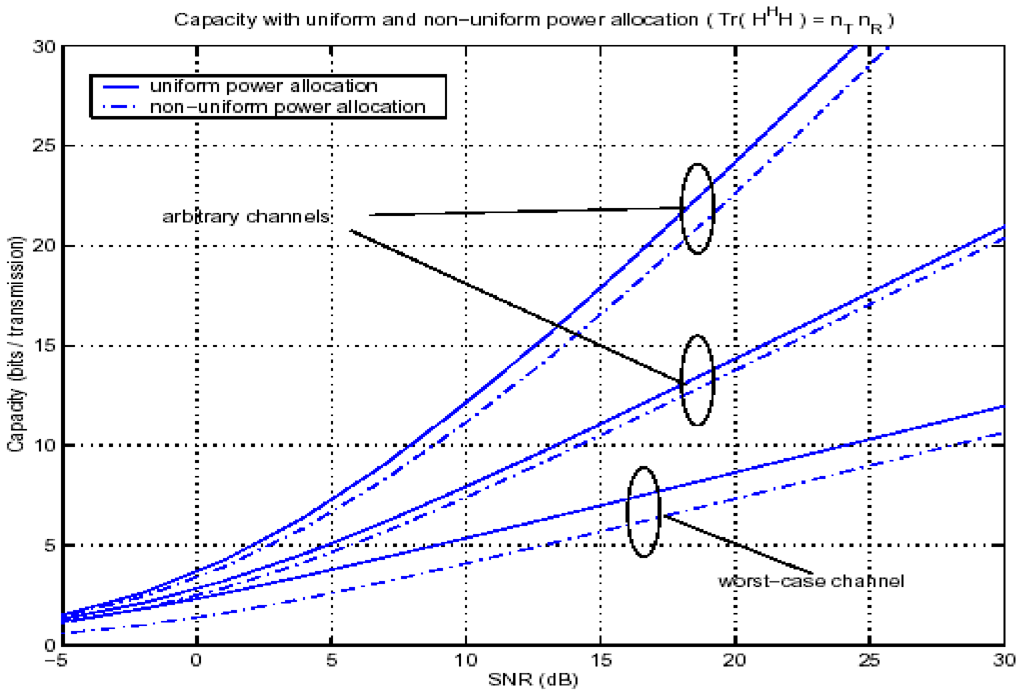
Nash equilibrium on every sub-game



Always the same result: UPA



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OUTAGE Capacity

Ergodic capacity has no sense for some channels.

$$C_{outage} = Capacity\ achieved\ with\ probability\ greater\ than\ \varepsilon$$

Formulation of the problem: Discrete set of N channels

$$t_i = \begin{cases} 1 & outage \quad con \quad \underline{\underline{H}}_i \\ 0 & no\ outage \quad con \quad \underline{\underline{H}}_i \end{cases}$$

Define outage for every channel realization

Difficult to solve due to the finite set 0/1

$$\min_{\rho_i, t_i} \sum_{i=1}^{N} \rho_i . t_i$$

$$s.t.$$

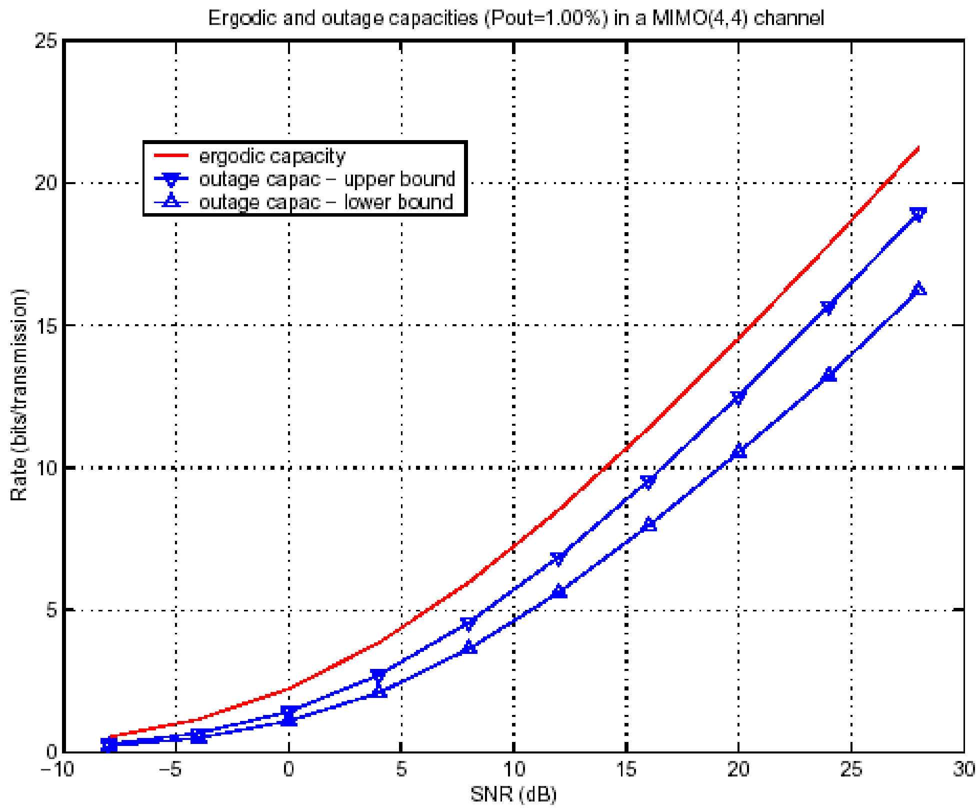
$$\rho_i = \ln\left(\det\left[\underline{I} + \underline{\underline{H}}_i . \underline{\underline{Q}} . \underline{\underline{H}}_i^H\right]\right) \geq C_{target} . (1 - t_i)$$

$$Trace\left(\underline{\underline{Q}}\right) \leq E_T \quad t_i \in (0,1)$$

Solve for continous t_i (convex problem)

LOWER BOUND: The solution to the continous problem.

UPPER BOUND: Use the lower bound solution in the discrete problem to find out the t_i



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Antenna Selection

$$C(1) = Ln\left[1 + \left(\underline{h}_1^H . \underline{h}_1\right)\frac{E_T}{N_0}\right]$$

Add Tx antenna following this iteration

$$\underline{\underline{Q}}.\left(\underline{\underline{H}}_m^H . \underline{\underline{H}}_m + \underline{h}_{m+1} . \underline{h}_{m+1}^H\right) = \underline{\underline{Q}}.\left(\sum_{q=1}^{m} \underline{h}_q . \underline{h}_q^H + \underline{h}_{m+1} . \underline{h}_{m+1}^H\right)$$

$$\underline{\underline{A}} = \underline{\underline{I}}_{n_T} + \underline{\underline{Q}}.\sum_{q=1}^{m} \underline{h}_q . \underline{h}_q^H$$

and

$$\underline{\underline{B}} = \underline{\underline{Q}}.\left(\underline{h}_{m+1} . \underline{h}_{m+1}^H\right)$$

$$\det\left[\underline{\underline{A}} + \underline{\underline{B}}\right] = \det\left[\underline{\underline{A}}\right].\det\left[\underline{\underline{I}} + \underline{\underline{A}}^{-1} . \underline{\underline{B}}\right]$$

$$C(m+1) = C(m) + Ln\left[1 + \left(\underline{h}_{m+1} . \left(\underline{\underline{I}}_{n_T} + \underline{\underline{Q}}.\sum_{q=1}^{m} \underline{h}_q . \underline{h}_q^H\right)^{-1} . \underline{\underline{Q}}\underline{h}_{m+1}^H\right)\right]$$

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A gateway to advanced communication technologies

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