

innovating communications

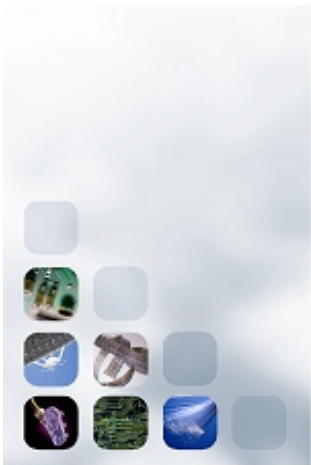
## The Centre Tecnològic de Telecomunicacions de Catalunya

*A gateway to advanced communication technologies*

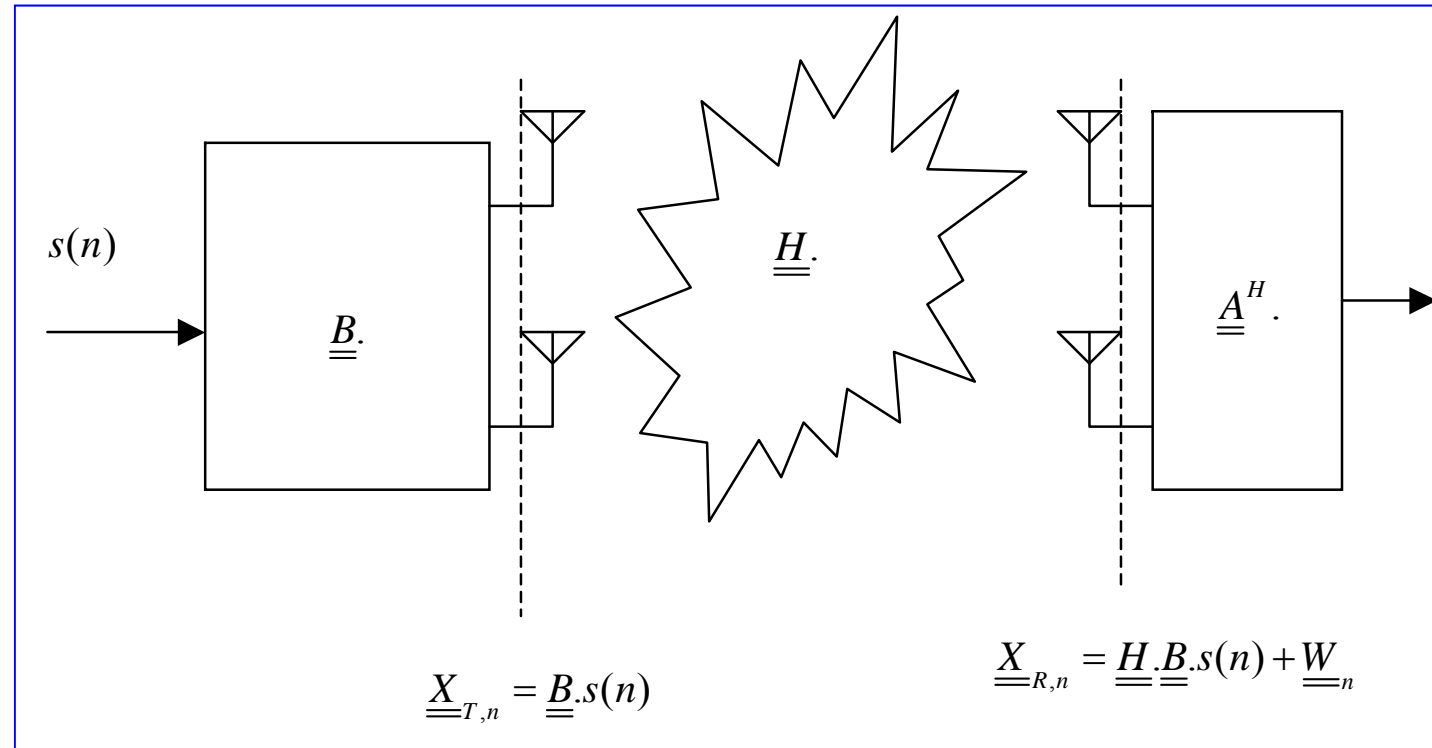
## MIMO2: Capacity

Miguel Ángel Lagunas

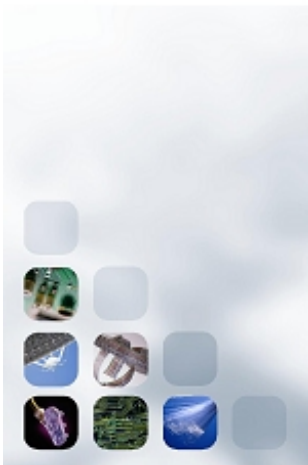
[m.a.lagunas@cttc.es](mailto:m.a.lagunas@cttc.es)



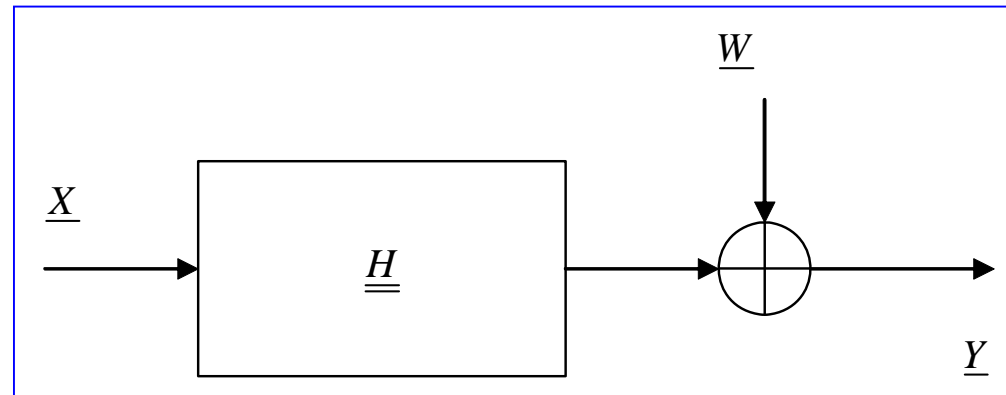
## When RATE is important



$$P_e = Q \left( \sqrt{\frac{\rho E_T}{N_0} \cdot \lambda_{\max}(\underline{\underline{R}}_H) \cdot \left( \frac{3}{2^{n_s} - 1} \right)} \right)$$



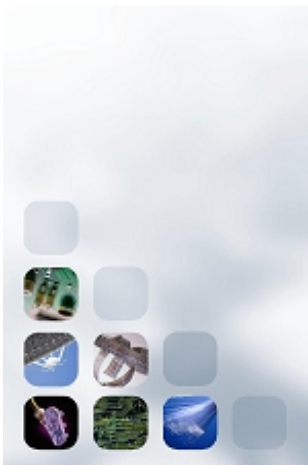
## ***Communications is a problem of MUTUAL INFORMATION***



The mutual information between two random vectors is:

$$I(\underline{X}, \underline{Y}) = \iint f(\underline{X}, \underline{Y}) \cdot \text{Ln} \left[ \frac{f(\underline{X}, \underline{Y})}{f(\underline{X}) \cdot f(\underline{Y})} \right] \cdot d\underline{X} \cdot d\underline{Y}$$

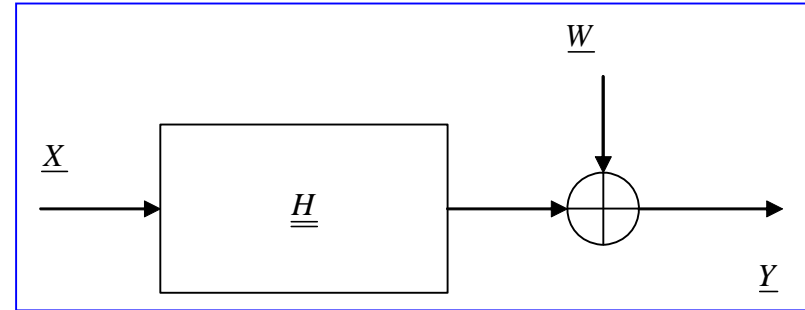
Maximize  $I(\underline{X}, \underline{Y})$  with respect noise and channel  
ONLY Tx has something to do with it!!!



## Maximization: First step

Assume that:

$$\underline{X} = G(\underline{0}, \underline{Q}) \quad \underline{W} = G(\underline{0}, \underline{R}_0)$$



### 1.-Gaussian Coding: The first step to maximize $I(X, Y)$

$$\max_{f(\cdot)} \int f(x) \cdot \ln(f(x)) \cdot dx$$

$$\int f(x) \cdot dx = 1$$

$$\int x^2 \cdot f(x) \cdot dx = \sigma_x^2$$



$$f(x) = k \cdot e^{-\frac{x^2}{2 \cdot \sigma^2}}$$

## Second steep\_ Pass I : The Importance of the Determinant of the Tx Covariance

The input distribution and Tx power

$$\Pr(\underline{X}) = \det(\underline{Q}^{-1}) \cdot \exp\left\{-\left(\underline{X}^H \cdot \underline{Q} \cdot \underline{X}\right)\right\}$$

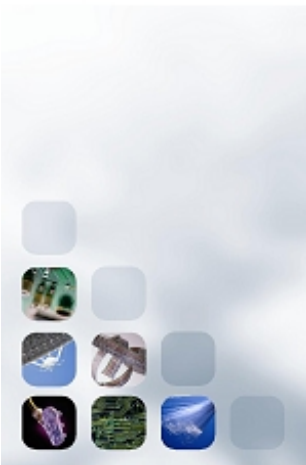
Two terms for the Mutual Information

Joint distribution

$$I = \iint f(\underline{X}, \underline{Y}) \cdot \text{Ln} \left[ \frac{\det[\underline{Q}] \cdot \det[\underline{R}_{yy}]}{\det \begin{pmatrix} \underline{Q} & \underline{R}_{xy} \\ \underline{R}_{yx} & \underline{R}_{yy} \end{pmatrix}} \right] \cdot d\underline{X} \cdot d\underline{Y} +$$

$$- \iint f(\underline{X}, \underline{Y}) \cdot \left[ \underline{X}^H \cdot \underline{R}_{xy} \cdot \underline{Y} + \underline{Y}^H \cdot \underline{R}_{yx} \cdot \underline{X} \right] \cdot d\underline{X} \cdot d\underline{Y}$$

Cross-terms equal to zero



$$I(\underline{X}, \underline{Y}) = Ln \left[ \frac{\det[\underline{Q}] \cdot \det[\underline{R}_{yy}]}{\det \begin{pmatrix} \underline{Q} & \underline{R}_{xy} \\ \underline{R}_{yx} & \underline{R}_{yy} \end{pmatrix}} \right]$$

Using the  
determinant rule  
(Schur)

$$\det \begin{pmatrix} \underline{Q} & \underline{R}_{xy} \\ \underline{R}_{yx} & \underline{R}_{yy} \end{pmatrix} = \det[\underline{Q}] \cdot \det \left[ \underline{R}_{yy} - \underline{R}_{yx} \cdot \underline{Q}^{-1} \cdot \underline{R}_{xy} \right]$$

$$C = Ln \left[ \frac{\det[\underline{R}_{yy}]}{\det \left[ \underline{R}_{yy} - \underline{R}_{yx} \cdot \underline{Q}^{-1} \cdot \underline{R}_{xy} \right]} \right]$$

Received Signal

Received Noise  
plus Interference

## MI in terms of channel, noise and Tx covariance

$$C = \text{Ln} \left[ \frac{\det \left[ \underline{\underline{R}}_{yy} \right]}{\det \left[ \underline{\underline{R}}_{yy} - \underline{\underline{R}}_{yx} \cdot \underline{\underline{Q}}^{-1} \cdot \underline{\underline{R}}_{xy} \right]} \right]$$

with

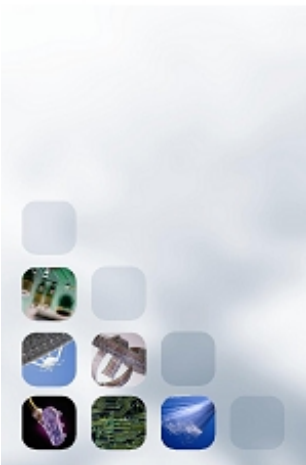
$$\underline{\underline{R}}_{yy} = \underline{\underline{H}} \cdot \underline{\underline{Q}} \cdot \underline{\underline{H}}^H + \underline{\underline{R}}_0$$

$$\underline{\underline{R}}_{xy} = \underline{\underline{Q}} \cdot \underline{\underline{H}}^H$$

$$\underline{\underline{R}}_{yx} = \underline{\underline{H}} \cdot \underline{\underline{Q}}$$

We denote MI as Capacity C but it is not strictly capacity

$$C = \text{Ln} \left[ \frac{\det \left[ \underline{\underline{R}}_0 + \underline{\underline{H}} \cdot \underline{\underline{Q}} \cdot \underline{\underline{H}}^H \right]}{\det \left[ \underline{\underline{R}}_0 \right]} \right] = \text{Ln} \left( \det \left[ \underline{\underline{I}}_{n_R} + \underline{\underline{R}}_0^{-1} \underline{\underline{H}} \cdot \underline{\underline{Q}} \cdot \underline{\underline{H}}^H \right] \right)$$





## *MI: Engineering terms*

$$C = \text{Ln} \left( \det \left[ \underset{=n_R}{\underline{\underline{I}}} + \underset{=0}{\underline{\underline{R}}}^{-1} \underset{=}{\underline{\underline{H}}} \cdot \underset{=}{\underline{\underline{Q}}} \cdot \underset{=}{\underline{\underline{H}}}^H \right] \right) = \text{Ln} \left( \det \left[ \underset{=n_R}{\underline{\underline{I}}} + \underset{=}{\underline{\underline{H}}}^H \underset{=0}{\underline{\underline{R}}}^{-1} \underset{=}{\underline{\underline{H}}} \cdot \underset{=}{\underline{\underline{Q}}} \right] \right)$$



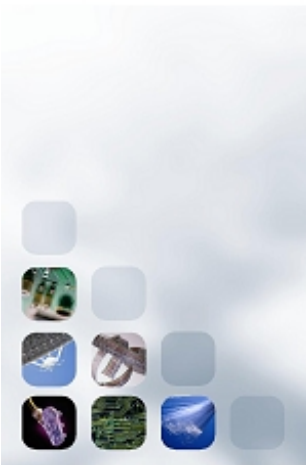
$$\underset{=H}{\underline{\underline{R}}} = \underset{=}{\underline{\underline{H}}}^H \cdot \underset{=0}{\underline{\underline{R}}}^{-1} \cdot \underset{=}{\underline{\underline{H}}}$$

The channel SNR

$$\underset{=H}{\underline{\underline{R}}} \cdot \underset{=}{\underline{\underline{Q}}} = \underset{=}{\underline{\underline{H}}}^H \cdot \underset{=0}{\underline{\underline{R}}}^{-1} \cdot \underset{=}{\underline{\underline{H}}} \cdot \underset{=}{\underline{\underline{Q}}}$$

The absolute SNR of  
the communications  
scenario

Let us see some examples for MIMO channels





## *MI (Capacity): Some Examples*

### Capacity for Uniform Power Allocation UPA

$$C = \text{Ln} \left( \det \left[ \underset{=n_R}{I} + \frac{\gamma}{n_T} \underline{\underline{H}} \cdot \underline{\underline{H}}^H \right] \right)$$

with

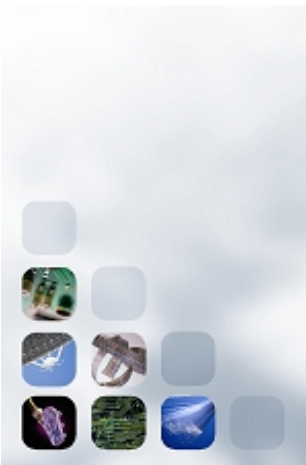
$$\gamma = \frac{E_T}{N_0}$$

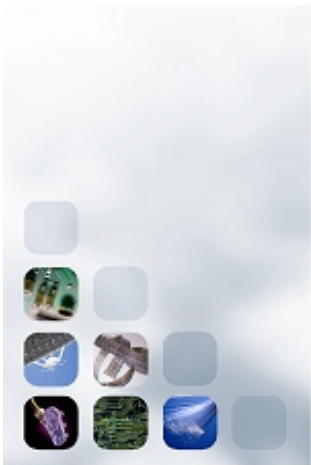
Normaliced with the Froebenius norm of the trace

$$C_{SISO} = \text{Ln} [1 + \gamma]$$

$$C_{MISO} = \text{Ln} \left[ 1 + \frac{\gamma}{n_T} \underline{h}^H \cdot \underline{h} \right]$$

$$C_{SIMO} = \text{Ln} \left[ 1 + \gamma \cdot \underline{h}^H \cdot \underline{h} \right]$$





No advantage for capacity derived for increasing either the Tx number of antennas (MISO) or the Rx number of antennas (SIMO)

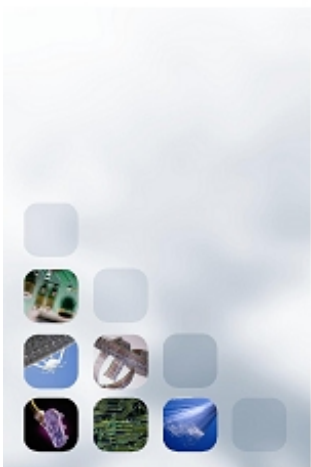
$$C_{MISO} = \text{Ln} \left[ 1 + \frac{\gamma}{n_T} \underline{h}^H \cdot \underline{h} \right]$$

$$C_{SIMO} = \text{Ln} \left[ 1 + \gamma \cdot \underline{h}^H \cdot \underline{h} \right]$$

$$C_{MISO} = C_{SIMO} = \text{Ln} [1 + \gamma]$$

$$C_{MISO} = \text{Ln} [1 + \gamma]$$

$$C_{SIMO} = \text{Ln} [1 + \gamma \cdot n_R]$$



## ***MIMO Increase linearly the channel capacity***

Assuming UPA for the channel eigenmodes, the mutual information is (sub-optimum):

$$C = \sum_{q=1}^{\min(n_T, n_R)} \text{Ln} \left[ 1 + \frac{E_T}{N_0 \cdot n_T} \cdot \lambda_q \right]$$

And, for uniform channel eigenvalues (gain), MI scales linearly with the minimum number of antennas among Tx/Rx (Worst channel case).

$$C = \min(n_T, n_R) \cdot \text{Ln} \left[ 1 + \frac{E_T \cdot \lambda}{n_T \cdot N_0} \right]$$

## Achieving capacity: Step 2.- Optimal Tx architecture

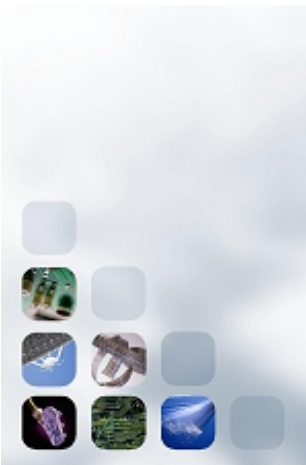
$$C = \text{Ln} \left( \det \left[ \underline{I}_{=n_T} + \underline{Q} \cdot \underline{H}^H \underline{R}_{=0}^{-1} \underline{H} \right] \right) = \text{Ln} \left( \det \left[ \underline{I}_{=n_R} + \underline{H}^H \underline{R}_{=0}^{-1} \underline{H} \cdot \underline{Q} \right] \right) =$$

$$C = \text{Ln} \left( \det \left[ \underline{I}_{=n_T} + \underline{Q} \cdot \underline{R}_{=H} \right] \right)$$

Since, for definite positive matrixes

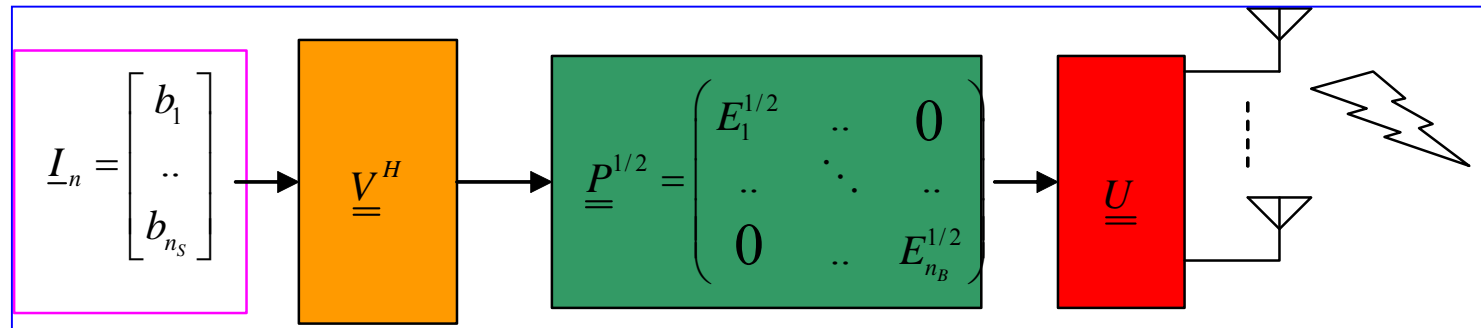
$$\det(\underline{A}) \leq \prod A_{ii}$$

The optimal Tx strategy is to diagonalize the channel matrix



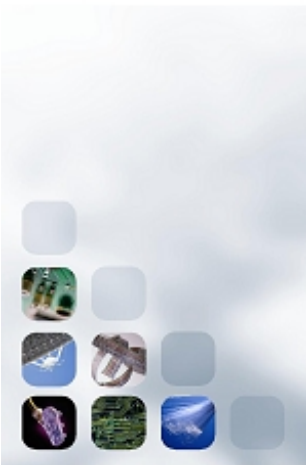
With the svd of the channel matrix

$$\underline{\underline{R}}_H = \underline{\underline{U}} \cdot \underline{\underline{L}} \cdot \underline{\underline{U}}^H = \begin{bmatrix} \underline{u}_{\max} & \dots & \underline{u}_{\min} \end{bmatrix} \begin{bmatrix} \lambda_{\max} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \lambda_{\min} \end{bmatrix} \begin{bmatrix} \underline{u}_{\max}^H \\ \dots \\ \underline{u}_{\min}^H \end{bmatrix}$$



$$\underline{\underline{Q}} = \underline{\underline{B}} \cdot \underline{\underline{B}}^H = \left( \underline{\underline{U}} \cdot \underline{\underline{P}}^{1/2} \cdot \underline{\underline{V}}^H \right) \cdot \left( \underline{\underline{U}} \cdot \underline{\underline{P}}^{1/2} \cdot \underline{\underline{V}}^H \right)^H = \underline{\underline{U}} \cdot \underline{\underline{P}} \cdot \underline{\underline{U}}^H$$

MI independent of matrix  $\underline{\underline{V}}$ , only beamforming as the eigenvectors of  $\underline{\underline{R}}_H$  is required



$$I(X, Y) = \text{Ln} \left( \det \left[ \underline{I}_{=n_T} + \frac{1}{N_0} \underline{U}_{=H} \cdot \underline{P}^{1/2} \cdot \underline{V}^H \cdot \underline{V} \cdot \underline{P}^{1/2} \cdot \underline{D} \cdot \underline{U}_{=H}^H \right] \right) =$$

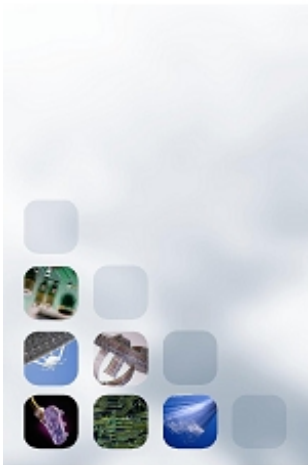
$$I(X, Y) = \text{Ln} \left( \det \left[ \underline{I}_{=\min(n_T, n_R)} + \frac{1}{N_0} \underline{P}^{1/2} \cdot \underline{V}^H \cdot \underline{V} \cdot \underline{P}^{1/2} \cdot \underline{D}_{=H} \right] \right)$$

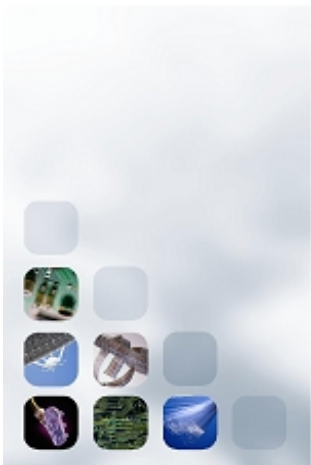


$$\underline{V}^H \cdot \underline{V} = \underline{I}_{=n_T} \quad \underline{Q} = \underline{U} \cdot \underline{Z} \cdot \underline{U}^H \quad \underline{R}_{=H} = \underline{U} \cdot \underline{L} \cdot \underline{U}^H$$



$$I(\underline{X}, \underline{Y}) = C = \ln \left[ \det \left( \underline{I} + \underline{Z} \cdot \underline{L} \right) \right]$$





## Capacity: Steep 3.- Optimum power allocation

$$C = \text{Ln} \left( \det \left[ \underset{= \min(n_T, n_R)}{I} + \frac{1}{N_0} \underset{=} {P^{1/2}} \cdot \underset{=} {V^H} \cdot \underset{=} {V} \cdot \underset{=} {P^{1/2}} \cdot \underset{=} {D_H} \right] \right) =$$

$$= \sum_{q=1}^{\min(n_T, n_R)} \text{Ln} \left[ 1 + \frac{1}{N_0} z(q) \cdot \lambda_H(q) \right] \Big|_{MAX}$$

Constrained to  $\sum_{q=1}^{\min(n_T, n_R)} z(q) = E_T$       The Lagrangian is:

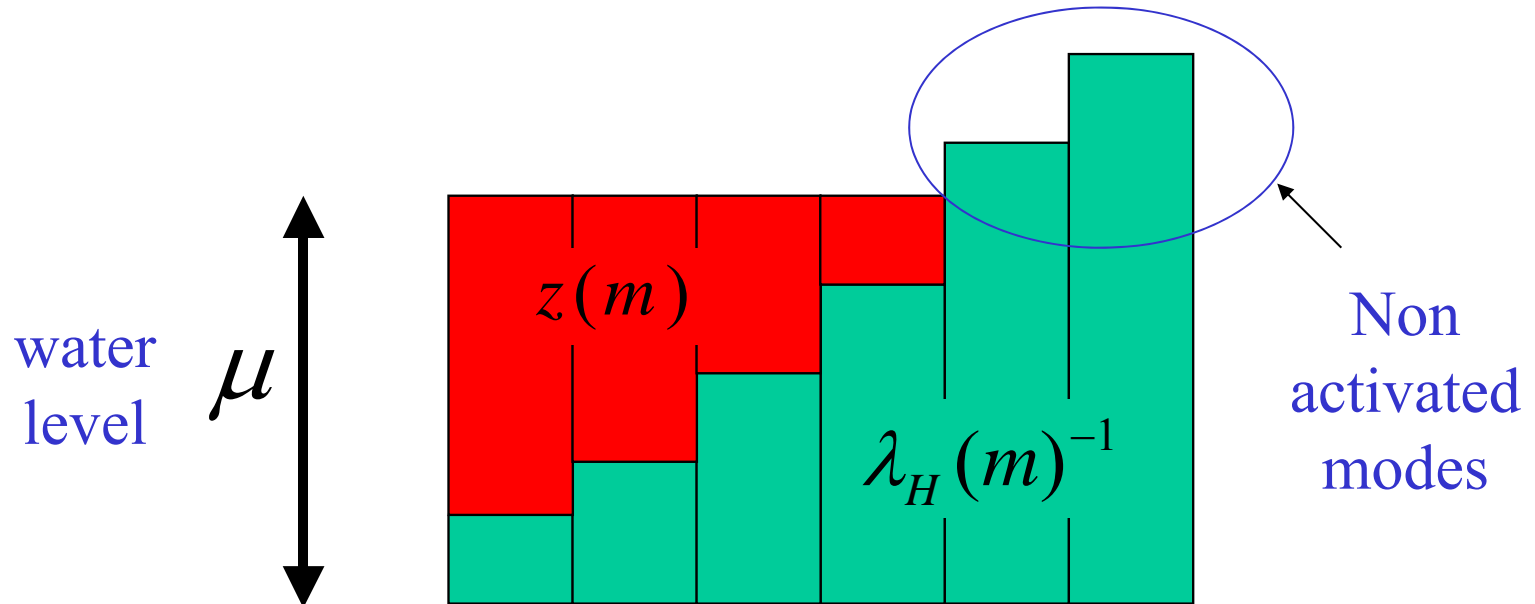
$$\Delta = \sum_{q=1}^{\min(n_T, n_R)} \text{Ln} \left[ 1 + \frac{1}{N_0} z(q) \cdot \lambda_H(q) \right] - m1 \cdot \left( \sum_{q=1}^{\min(n_T, n_R)} z(q) = E_T \right)$$

$$\partial \Delta = 0 = \frac{1}{N_0} \frac{\lambda_H(q)}{1 + z(q) \cdot \lambda_H(q)} - m1 \Rightarrow z(q) = \left[ \mu - \frac{1}{\lambda_H(q)} \right]^+$$

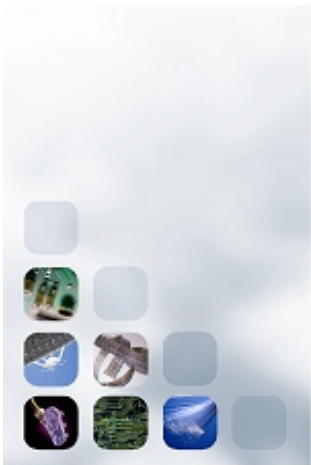




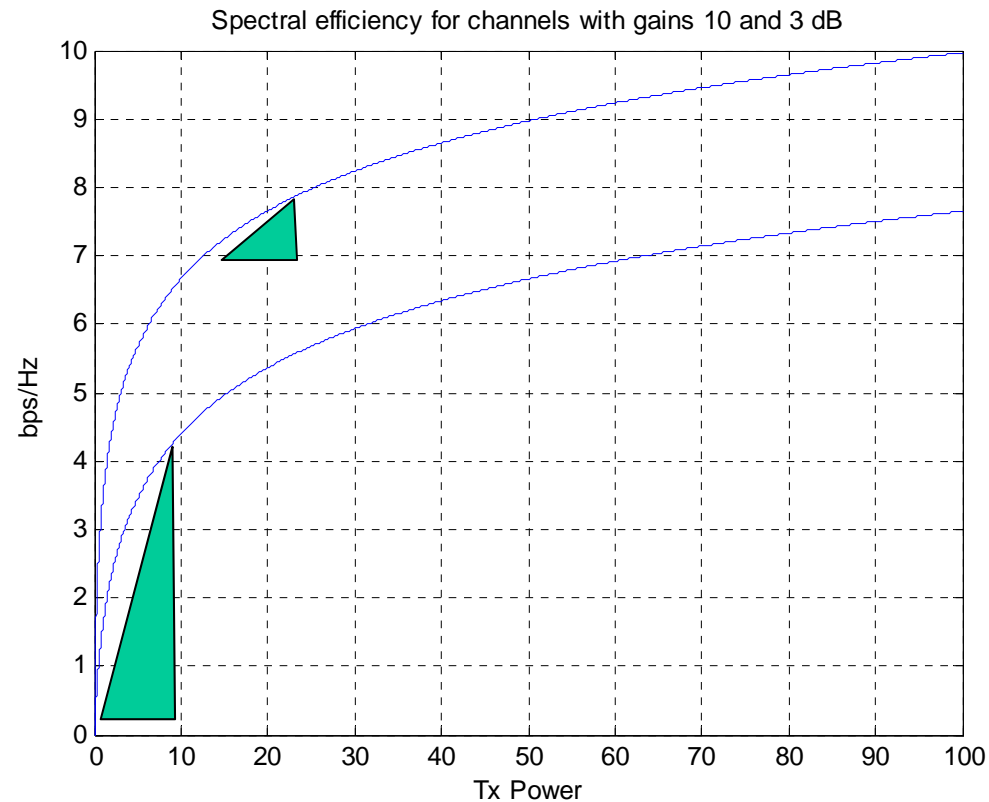
## Water-filling algorithm



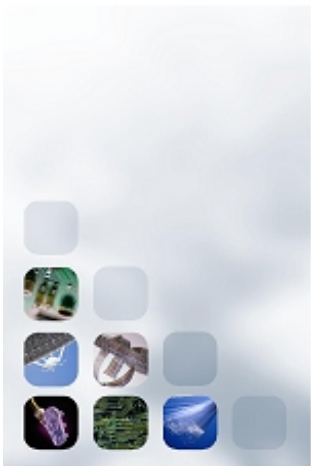
- Note that not all the modes are used
- Fairness issue: More energy for the best modes
- Gaussian coding and bit-loading required



## *An intuitive explanation*



Due to saturation of  $\log(\cdot)$  it is more rewarding to start allocating power to the lower channel



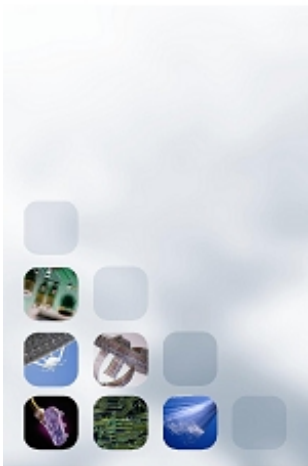
## Capacity for UPA

When UPA is used the capacity is:

$$C = \ln \left( \det \left[ \underline{I} + \frac{E_T}{n_T} \underline{R}_H \right] \right) = \sum_{q=1}^{\min(n_T, n_R)} \ln \left( 1 + \frac{E_T \cdot \lambda_H(q)}{n_T} \right)$$

There is an interesting approximation of this expression for moderate/high SNR regime, showing that the geometric mean of the eigenvalues determines the UPA capacity:

$$C = \ln \left( \prod_{q=1}^{n_M \triangleright \min(n_T, n_R)} \left( 1 + \frac{E_T \cdot \lambda_H(q)}{n_T} \right) \right) \approx n_R \cdot \ln \left( \frac{E_T \cdot \lambda_{GEO}}{n_T} \right)$$



## *Instantaneous Max. Capacity*

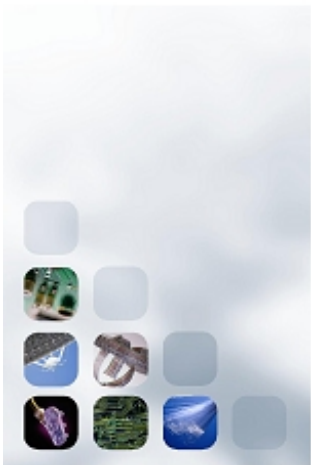
When all the modes are activated:

$$\mu = \frac{E_T}{n_M} + \lambda_{HARM} \quad \text{where} \quad \lambda_{HARM} = \frac{n_M}{\sum_{q=1}^{n_M} \lambda_H^{-1}(q)}$$

The power allocation is:

$$z(q) = \frac{E_T}{n_M} + \lambda_{HARM}^{-1} - \lambda_H^{-1}(q)$$

Note that all the modes are activated when the mean energy per mode is greater than the difference between the inverse of the harmonic mean and the inverse of the minimum eigenmode gain.



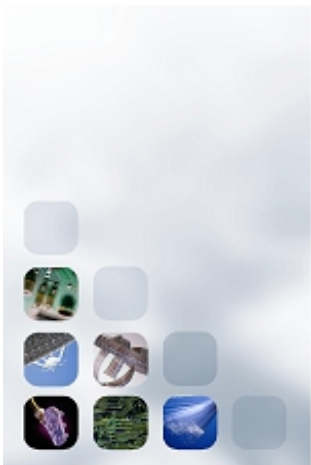
$$C = n_M \cdot \ln \left( \left[ \frac{E_T \cdot \lambda_{GEO}}{n_M} \right] + \left[ \frac{\lambda_{GEO}}{\lambda_{HARM}} \right] \right)$$

UPA contribution

(No CSIT required)

Gain derived  
from CSIT

- The gain derived from CSIT does not depend on the power budget.
- For large SNR regimes the optimum power allocation to achieve capacity is UPA.
- DSL services. Not power limited. No mask constraints (twisted pairs). UPA



## Beamforming to achieve capacity

$$\mu = E_T + \frac{1}{\lambda_{H,\max}}; z(0) = E_T$$

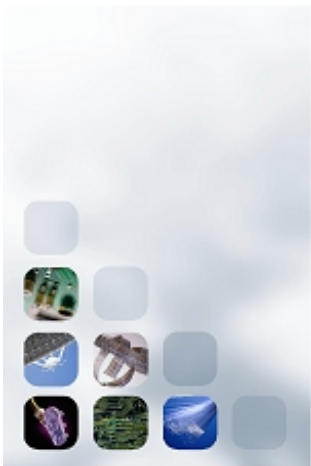
$$C_{\text{BEAMFORMING}} = \text{Ln} \left[ 1 + \frac{E_T}{N_0} \cdot \lambda_{H,\max} \right]$$

When only the maximum gain mode is activated the optimum policy is beamforming

This is the case when the second and the first eigenmode gains satisfy:

$$\lambda_2 \leq \frac{1}{E_T + \left(\frac{1}{\lambda_1}\right)} \Rightarrow E_T < \left(\frac{1}{\lambda_2}\right) - \left(\frac{1}{\lambda_1}\right)$$

Maximum quality and maximum rate policies coincide



## Example:

$$\underline{\underline{H}} \quad \underline{\underline{H}}$$

$$\begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 8 & 0 \\ 0 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 2 \\ 2 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 7 \\ 7 & 5 \end{pmatrix}$$

*MaxEigenvalue*

10 (10dB)

8 (9dB)

10.70 (10.29dB)

14.93 (11.7dB)

Determinant

50 (5.67bps / Hz.)

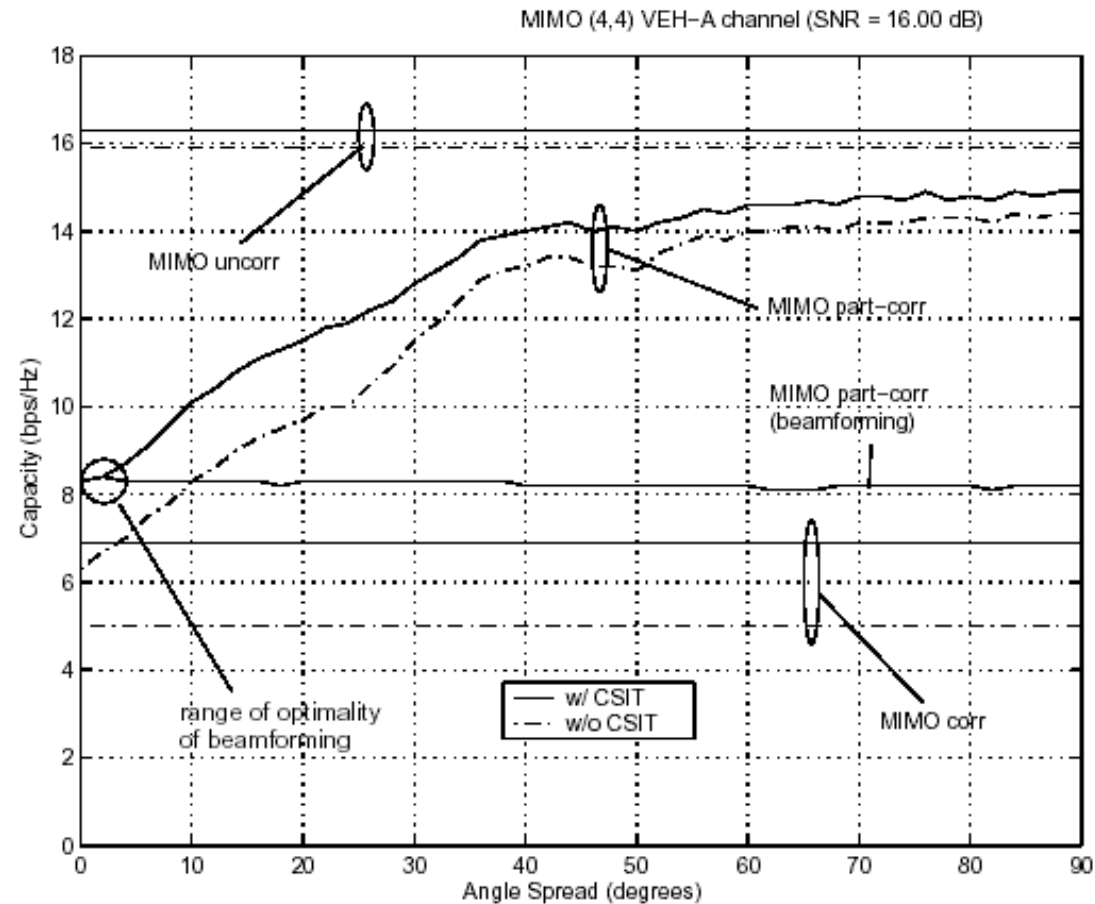
15 (5.83bps / Hz.)

46 (5.52bps / Hz.)

1 (2bps / Hz.)



## Correlation degrades the geometric mean (the channel determinant)

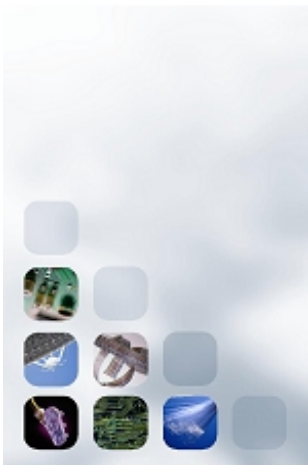


## ***Ergodic Capacity***

When there is not CSIT, the Tx design depends on the statistical knowledge of the channel and the instantaneous capacity is averaged over channel statistics.

$$\underline{\underline{Q}} = \max_{\underline{\underline{Q}}} \left( E \left[ \ln \left( \det \left( \underline{\underline{I}} + \underline{\underline{R}}_H \cdot \underline{\underline{Q}} \right) \right) \right] \right)$$

with  $\text{Trace}(\underline{\underline{Q}}) = E_T$       Is the problem to be solved

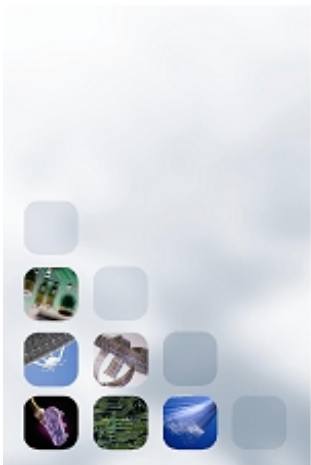


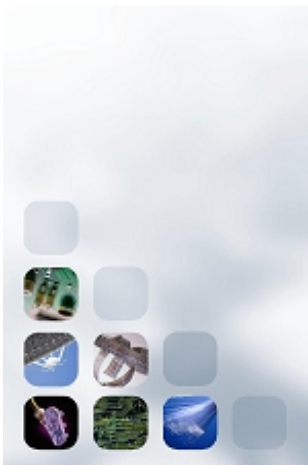


The solution to the previous problem for channel matrixes with i.i.d. entries following a pdf  $p(x)$  such that it is symmetric along the origin, i.e.  $p(x)=p(-x)$ , and circular symmetry  $p(x)=p(x.e^{j\phi})$  is UPA (Uniform Power Allocation)



The optimum policy for maximizing ergodic capacity is UPA

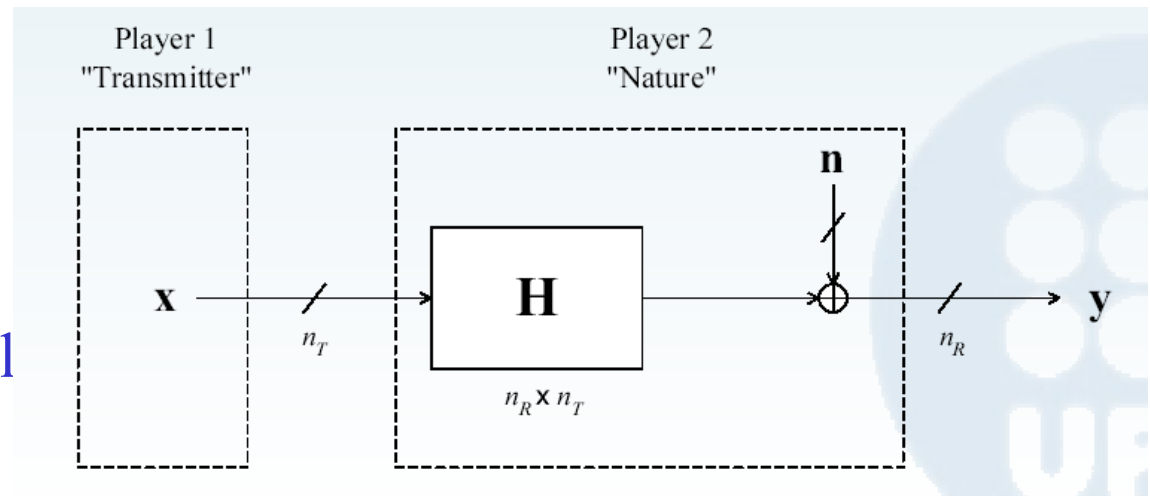




## GAME THEORY APPROACH

- Strategic game with pure strategies
- Strategic game with mixed pure strategies (random over pure strategies)
- Stackelberg game: Strategies depends on the previous selected strategy by another player)

Solution  
independent  
of the channel  
constrains





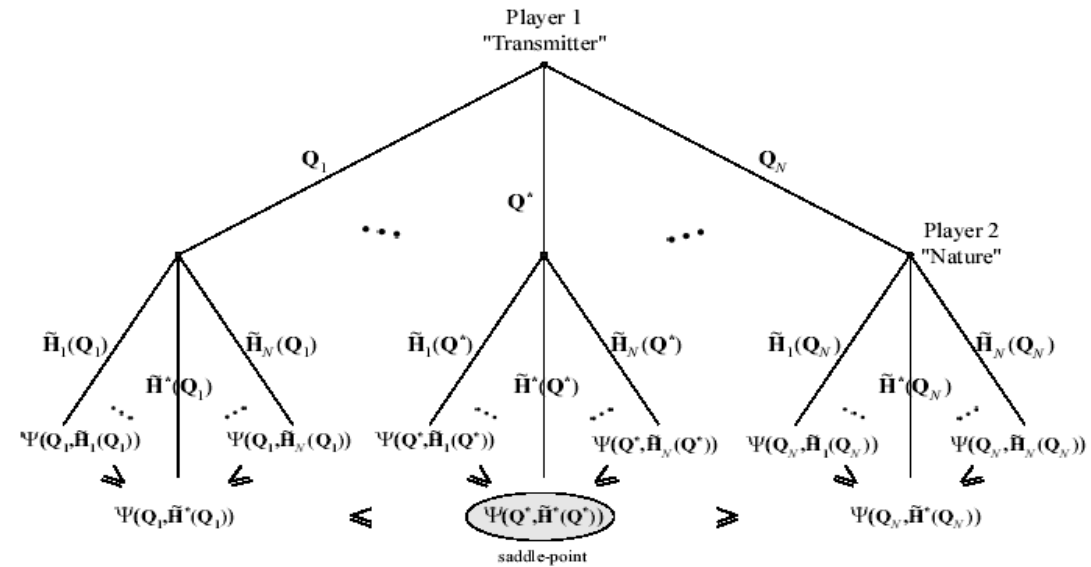
		Player 2 "Nature"				
		$\tilde{H}_1$	...	$\tilde{H}^*$	...	$\tilde{H}_N$
Player 1 "Transmitter"	$Q_1$	$\Psi(Q_1, \tilde{H}_1)$	...	$\Psi(Q_1, \tilde{H}^*)$	...	$\Psi(Q_1, \tilde{H}_N)$
	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
	$Q^*$	$\Psi(Q^*, \tilde{H}_1)$	...	$\Psi(Q^*, \tilde{H}^*)$ saddle-point	...	$\Psi(Q^*, \tilde{H}_N)$
	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
	$Q_N$	$\Psi(Q_N, \tilde{H}_1)$	...	$\Psi(Q_N, \tilde{H}^*)$	...	$\Psi(Q_N, \tilde{H}_N)$

Payoff:  $\Psi(Q, \tilde{H})$

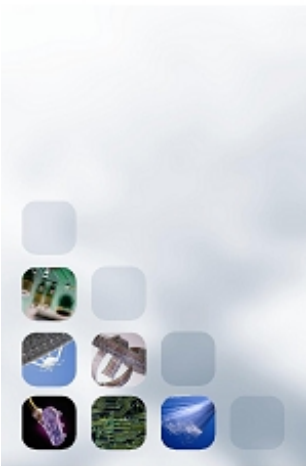
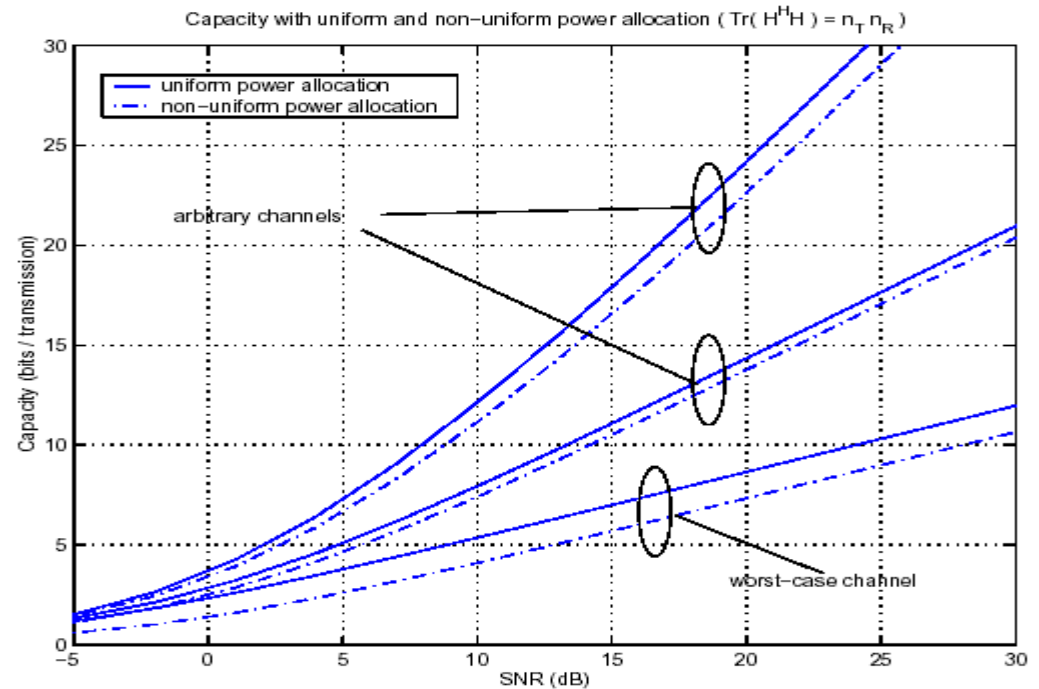
Pure strategies:

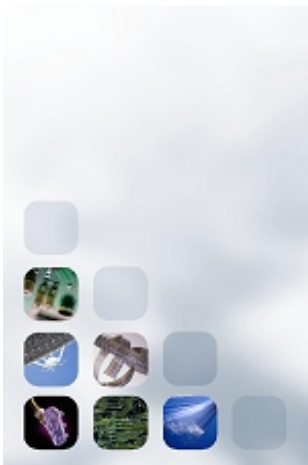
- Tx performs water-filling of channel
- Channel performs inverse waterfilling of Tx
- Nash equilibrium for flat Tx and Channel

# Nash equilibrium on every sub-game



Always the same result:  
UPA





## OUTAGE Capacity

Ergodic capacity has no sense for some channels.

$C_{outage}$  = Capacity achieved with probability greater than  $\varepsilon$

Formulation of the problem: Discrete set of N channels

$$t_i = \begin{cases} 1 & \text{outage con } \underline{\underline{H}}_i \\ 0 & \text{no outage con } \underline{\underline{H}}_i \end{cases}$$
 Define outage for every channel realization

Difficult to solve  
due to the finite  
set 0/1

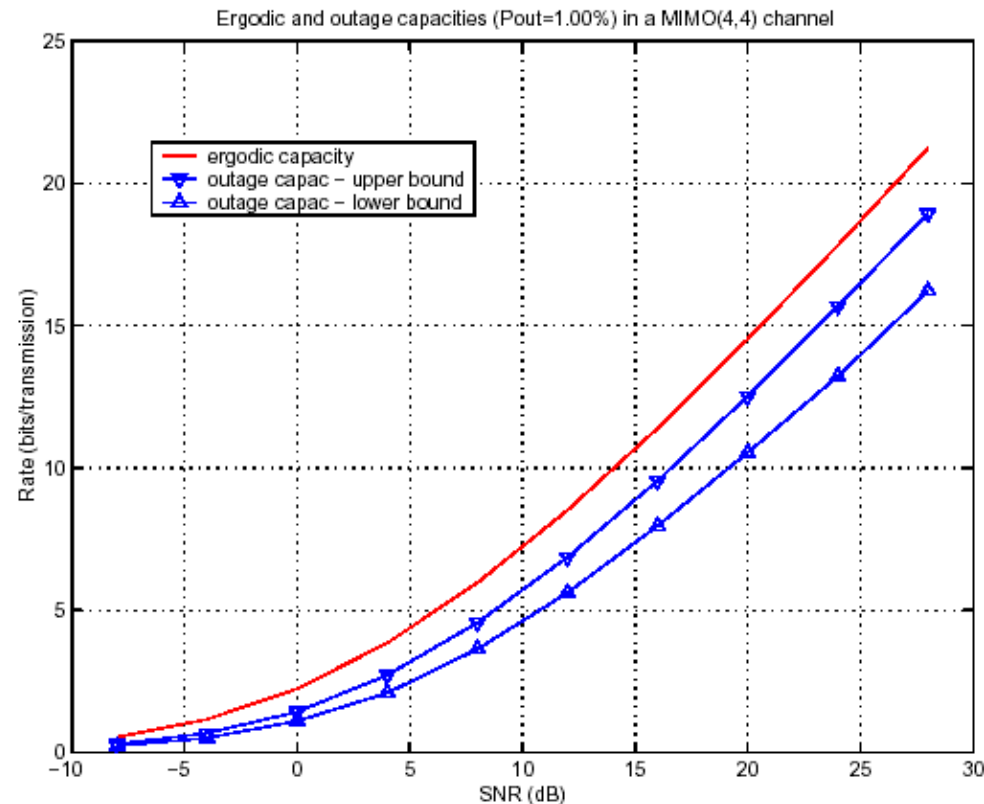
$$\begin{aligned}
 & \min_{\rho_i, t_i} \sum_{i=1}^N \rho_i \cdot t_i \\
 & s.t. \\
 & \rho_i = \ln \left( \det \left[ \underline{\underline{I}} + \underline{\underline{H}}_i \cdot \underline{\underline{Q}} \cdot \underline{\underline{H}}_i^H \right] \right) \geq C_{target} \cdot (1 - t_i) \\
 & \text{Trace}(\underline{\underline{Q}}) \leq E_T \quad t_i \in (0,1)
 \end{aligned}$$

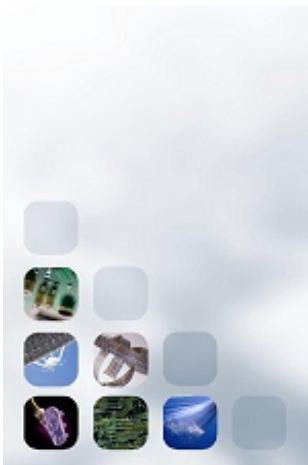


Solve for continuous  $t_i$  (convex problem)

LOWER BOUND: The solution to the continuous problem.

UPPER BOUND: Use the lower bound solution in the discrete problem to find out the  $t_i$





## Antenna Selection

$$C(1) = Ln \left[ 1 + \left( \underline{h}_1^H \cdot \underline{h}_1 \right) \frac{E_T}{N_0} \right]$$

Add Tx antenna following this iteration

$$\underline{\underline{Q}} \cdot \left( \underline{\underline{H}}_m^H \cdot \underline{\underline{H}}_m + \underline{h}_{m+1} \cdot \underline{h}_{m+1}^H \right) = \underline{\underline{Q}} \cdot \left( \sum_{q=1}^m \underline{h}_q \cdot \underline{h}_q^H + \underline{h}_{m+1} \cdot \underline{h}_{m+1}^H \right)$$


$$\underline{\underline{A}} = \underline{\underline{I}}_{n_T} + \underline{\underline{Q}} \cdot \sum_{q=1}^m \underline{h}_q \cdot \underline{h}_q^H$$

$$\det \left[ \underline{\underline{A}} + \underline{\underline{B}} \right] = \det \left[ \underline{\underline{A}} \right] \cdot \det \left[ \underline{\underline{I}} + \underline{\underline{A}}^{-1} \cdot \underline{\underline{B}} \right]$$

and

$$\underline{\underline{B}} = \underline{\underline{Q}} \cdot \left( \underline{h}_{m+1} \cdot \underline{h}_{m+1}^H \right)$$

$$C(m+1) = C(m) + Ln \left[ 1 + \left( \underline{h}_{m+1} \cdot \left( \underline{\underline{I}}_{n_T} + \underline{\underline{Q}} \cdot \sum_{q=1}^m \underline{h}_q \cdot \underline{h}_q^H \right)^{-1} \cdot \underline{\underline{Q}} \underline{h}_{m+1}^H \right) \right]$$



innovating communications

# The Centre Tecnològic de Telecomunicacions de Catalunya

*A gateway to advanced communication technologies*

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